Subnormal operators with nontrivial quasinormal extensions

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1. Introduction. PUTNAM in [2] developed some interesting properties of certain completely subnormal operators. The results were presented as a generalization of known results about shift operators. It is not clear in [2], however, how much of a generalization they are and which types of completely subnormal operators they apply to.

This note will show that the completely subnormal operators to which Putnam's Theorem 1 applies are quasinormal. This characterization will considerably simplify the proof of that theorem. We will also get an interesting equivalent form of Putnam's Theorem 2.

Our notation will be that of [2]. Let $[X, Y]=XY-YX$ for bounded linear operators X, Y. Recall that an operator T is quasinormal if $[T, T^*T]=0$. Quasinormal operators are always subnormal.

2. Results. Our first result characterizes those T referred to in Theorem 1 of [2].

Theorem 1. *Let T be a completely subnormal operator on a Hilbert space $\mathfrak{H}$ with minimal normal extension N on $\mathfrak{K}$. Let Q denote the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Then*

$$Q(N^*N) = (N^*N)Q \tag{1}$$

if and only if T is quasinormal.

Proof. Suppose that T, N, $\mathfrak{H}$, $\mathfrak{K}$, and Q are defined as in Theorem 1. Then relative to the decomposition $\mathfrak{K}=(\mathfrak{K}\ominus\mathfrak{H})\oplus\mathfrak{H}$ we have

$$N = \begin{bmatrix} A & 0 \\ B & T \end{bmatrix}, \tag{2}$$

and

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{3}$$

A trivial matrix computation shows that (1) holds if and only if $T^*B=0$. But if N is normal, then $[T^*, T]=BB^*$. Hence $T^*[T^*, T]=0$ if (1) holds and $T^*(T^*T)=$

$=(T^*T)T^*$ as desired. Conversely, if $T^*(T^*T)=(T^*T)T^*$, then $T^*BB^*=0$. Hence $T^*B=0$ and (1) holds.

If T is quasinormal, then Brown has shown [1] that T can be written as

$$T=\begin{bmatrix} 0 & 0 & 0 & . \\ P & 0 & 0 & . \\ 0 & P & 0 & . \\ . & . & . & . \end{bmatrix} \tag{4}$$

where P is a positive operator. But then the block form (2) of N is

$$N=\left[\begin{array}{cccc|cccc} . & . & . & . & & & & \\ . & 0 & 0 & 0 & & & & \\ . & P & 0 & 0 & . & . & & \\ . & 0 & P & 0 & 0 & . & & \\ \hline . & 0 & 0 & P & 0 & 0 & 0 & . \\ . & 0 & 0 & 0 & P & 0 & 0 & . \\ . & 0 & 0 & 0 & 0 & P & 0 & . \\ . & . & . & . & . & . & . & . \end{array}\right] \tag{5}$$

provided T is one to one, which it is if it is completely subnormal.

As an immediate consequence we get that if T is completely subnormal and satisfies (1), then the unitary operator in its polar form *is* a bilateral shift and (1.4) of [2] follows immediately.

Using Theorem 1 and a slight modification of the proof of Theorem 2 in [2] we get:

Theorem 2. *Let T be a completely subnormal operator on $\mathfrak{H}$ with minimal normal extension N on $\mathfrak{K}$. Then either*

(i) *$\mathfrak{K}$ is the least subspace containing $\mathfrak{H}$ and invariant under N and N^*N, or*
(ii) *T has a non-normal quasinormal extension T_1 on $\mathfrak{H}_1\subseteqq\mathfrak{K}$. That is, there exists a non-trivial invariant subspace $\mathfrak{H}_1$ of N such that $\mathfrak{H}\subseteqq\mathfrak{H}_1$ and N restricted to $\mathfrak{H}_1$ is quasinormal.*

Furthermore, (i) *and* (ii) *cannot both be true for T.*

Proof. Suppose that T is a completely subnormal operator. As in [2] let $\mathfrak{H}_1$ denote the least subspace of $\mathfrak{K}$ containing $\mathfrak{H}$ and invariant under both N and N^*N. If $\mathfrak{H}_1=\mathfrak{K}$, then (i) holds. Suppose that $\mathfrak{H}_1\neq\mathfrak{K}$. Then by Theorem 1, the T_1 of [2, p. 114] is quasinormal so that (ii) holds.

That (i) and (ii) cannot both hold follows from the fact that if T_1 is quasinormal on $\mathfrak{H}_1$ with minimal normal extension N, then N and N^*N leave $\mathfrak{H}_1$ invariant. This is easily seen by observing that if N is given by (5), then N^*N is diagonal.

In [2], PUTNAM views condition (i) as being, in some sense, the opposite behavior from that exhibited by shifts. Our Theorem 2 shows exactly to what extent this is true. Theorem 2 also characterizes those completely subnormal operators with non-trivial quasinormal extensions.

3. An example. It is possible for a completely subnormal operator T to satisfy condition (ii) of Theorem 2 and not be quasinormal.

Example. Let T_1 be the quasinormal operator defined by the matrix (4) on $\mathfrak{h}=\sum_{i=0}^{\infty}\oplus\mathfrak{h}_i$, $\dim\mathfrak{h}_i=2$, with $P=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. Let $\mathfrak{M}_0$ be the subspace of $\mathfrak{h}_0$ spanned by $\begin{pmatrix}1\\0\end{pmatrix}$. Then $\mathfrak{M}=\mathfrak{M}_0\oplus\sum_{i=1}^{\infty}\oplus\mathfrak{h}_i$ is an invariant subspace for T_1 of codimension one. Let T be the restriction of T to $\mathfrak{M}$ so that T has the matrix

$$T=\begin{bmatrix}0&0&0&0&0&.\\1&0&0&0&0&.\\1&0&0&0&0&.\\0&1&1&0&0&.\\0&1&0&0&0&.\\0&0&0&1&1&.\\ .&.&.&.&.&.\end{bmatrix}.$$

Then T is not quasinormal since the (2,1) entry of $T(T^*T)$ is 2 while the (2,1) entry of $(T^*T)T$ is 3.

Similar examples can be constructed by taking a quasinormal T_1 in the form (4) on $\mathfrak{h}=\sum_{i=0}^{\infty}\oplus\mathfrak{h}_i$ and defining $\mathfrak{M}=\sum_{i=0}^{\infty}\oplus\mathfrak{M}_i$ where $\mathfrak{M}_i\subseteqq\mathfrak{h}_i$ and $P\mathfrak{M}_i\subseteqq\mathfrak{M}_{i+1}$.

In order to get T to not be quasinormal it is necessary to have some of the $\mathfrak{M}_i$ not be invariant subspaces for P. Care must be taken to guarantee that the minimal normal extension of T is also a normal extension of T_1. Note that P need not be positive in (4) for (4) to define a quasinormal operator. In fact, one only needs that P itself is quasinormal.

References

[1] ARLEN BROWN, On a class of operators, *Proc. Amer. Math. Soc.*, **4** (1953), 723—728.
[2] C. R. PUTNAM, Normal extensions of subnormal operators, *Acta Sci. Math.*, **36** (1974), 111—118.

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